SOLVING BASIC TRIG INEQUALITIES - Nghi Nguyen Method

(Authored by Nghi H Nguyen – Sept. 01, 2021)

THE TRIG UNIT CIRCLE AND BASIC TRIG FUNCTIONS

It is a circle with radius R = 1, and with an origin O. The unit circle defines the main trig functions of the variable arc x that varies and rotates counterclockwise on it.
On the unit circle, the value of the **arc x** is exactly **equal** to the corresponding **angle x**.
It is more convenient to use the arc x as the variable, instead of the angle x.
When the variable arc AM = x (in radians or degree) varies on the trig unit circle:

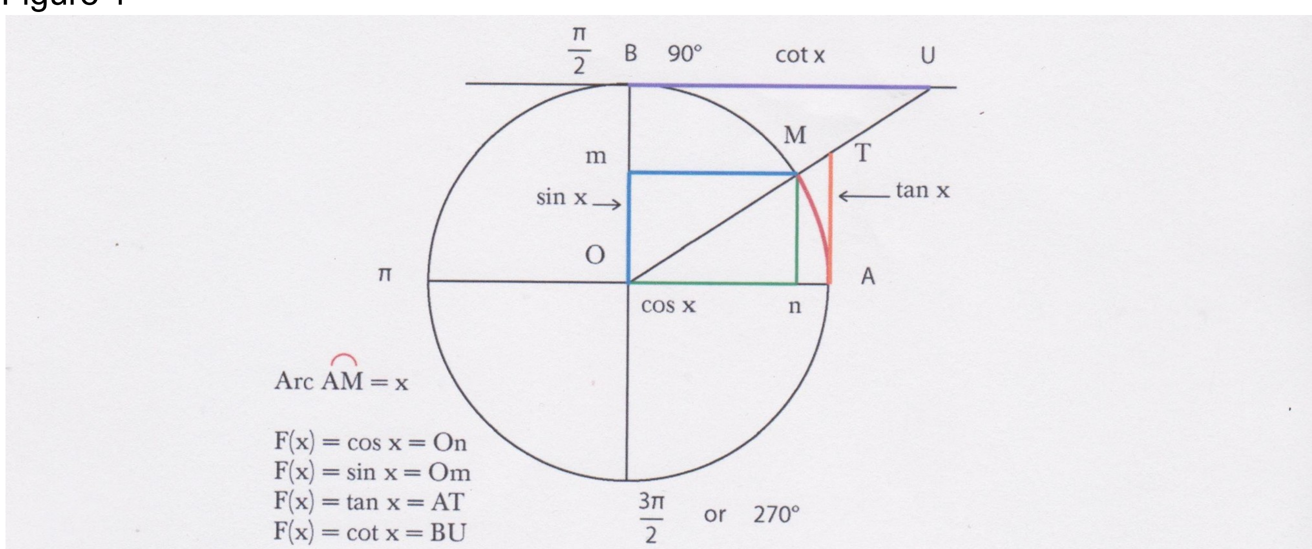
- The horizontal axis OAx defines the function **f(x) = cos x.** When the arc x varies from 0 to 2π, the function f(x) = cos x varies from 1 to -1, then, back to 1.

- The vertical axis OBy defines the function **f(x) = sin x**. When the arc x varies from 0 to 2π, the function f(x) = sin x varies from 0 to 1, then to -1, then back to 0.

- The vertical axis AT defines the function **f(x) = tan x.** When x varies from -π/2 to π/2, the function f(x) varies from - ∞ to +∞.

- The horizontal axis BU defines the function **f(x) = cot x**. When x varies from 0 to π, the function f(x) = cot x varies from +∞ to -∞.

Figure 1



SOLVING BASIC TRIG INEQUALITIES, OR SIMILAR

To solve a basic trig inequality, we use the trig unit circle and the 4 trig axis. The unit circle is numbered in radians or degrees. We also use trig calculators and trig tables (in trig books) and, specifically, the Trig Table of Special Arcs that gives specific values in radians (or degrees) of a few special arcs. Examples: sin $\pi/4$; cos $2\pi/3$; tan $\pi/3$….

Example 1. Solve $\sin x \le -1/2$ $(0, 2\pi)$

Solution. Transform the inequality into standard form: $f(x) = \sin x + 1/2 \le 0$

First, solve $f(x) = 0 \rightarrow \sin x = -1/2$. On the sin axis, take $\sin x = -1/2$

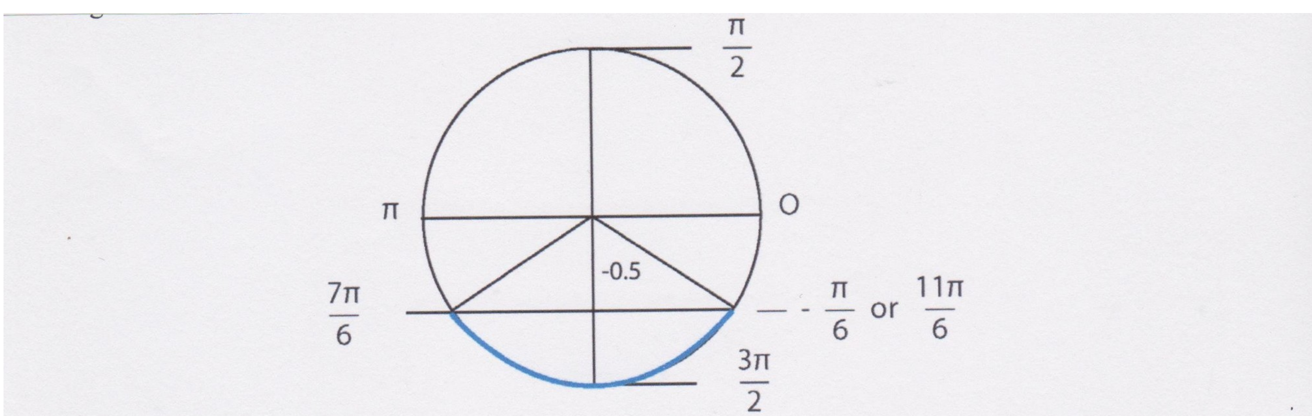
The unit circle and the trig Table of Special Arcs gives 2 end points at: $x1 = 7\pi/6$ and $x2 = 11\pi/6$ (or $-\pi/6$). Plot these 2 end points on the unit circle. They divide the circle into 2 arc lengths $(7\pi/6, 11\pi/6)$ and $(-\pi/6, 7\pi/6)$.

We can use the check point $(3\pi/2)$ to find the sign status of $f(x)$ inside this interval. We get: $f(x) = \sin(3\pi/2) + 1/2 = -1 + 1/2 = -1/2 < 0$.

The solution set of $f(x) \le 0$ is the closed interval **$[7\pi/6, 11\pi/6]$**. The 2 end points are included in the solution set.

Note. The above solution is the general method. In this case, we can visually solve the inequality much simpler. $\sin x < -1/2$ when x is inside the arc length $(7\pi/6, 11\pi/6)$.

Figure 2



Example 3. Solve $\sin x > 0.6$ $(0, 360^o)$

Solution. On the **sin axis**, take: $\sin x = 0.6$. Trig calculator gives: $x1 = 0.6 = \sin 36^o87$.
The unit circle gives another arc: $x2 = 0.6 = 180^o – 36^o87 = 143^o13$.
The visual solution set for $\sin x > 0.6$ is the arc length, or interval: (**36^o87, 143^o13**)
See Figure 3.

Example 4. Solve $\cos x \geq \sqrt{3}/2$

Solution. On the **cos axis**, plot the number $\sqrt{3}/2$. The trig Table of Special Arcs and the unit circle gives two solution arcs: $x = \pm \pi/6$.
We can see that the solution set, for $\cos x > 0.6$, is the arc length, or interval $(-\pi/6, \pi/6)$. The two end points $(\pi/6)$ and $(-\pi/6)$ are included in the solution set. Fig 4.

Figure 3

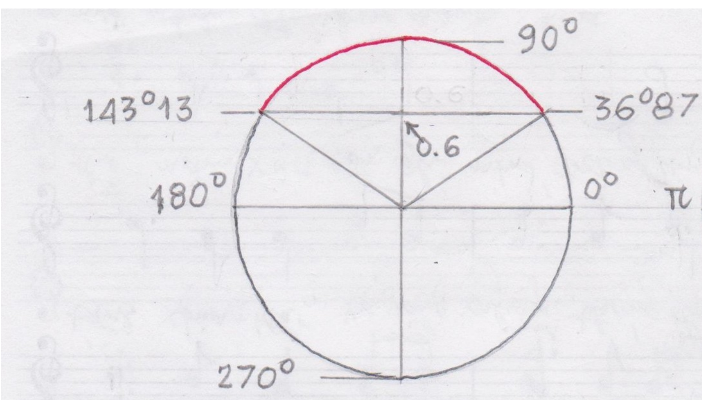
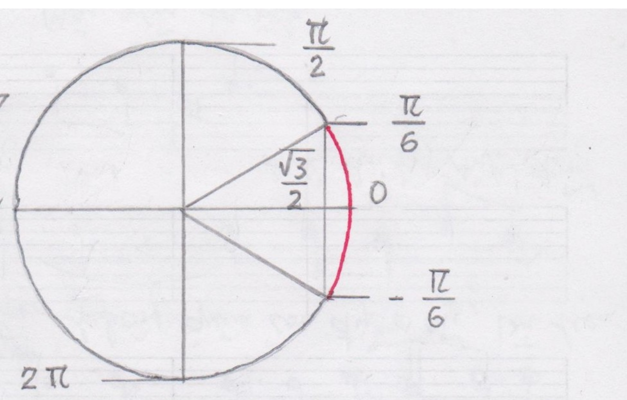


Figure 4



Example 5. Solve $\sin (x – \pi/4) < \sqrt{3}/2$ $(0, 2\pi)$

Solution. General form $f(x) = \sin (x – \pi/4) - \sqrt{3}/2 < 0$

First solve $f(x) = 0$ to get the 2 end points. We get: $\sin (x – \pi/4) = \sqrt{3}/2$. The unit circle and the trig Table of Special Arcs give 2 solutions:

1. $\sin (x- \pi/4) = \sin \pi/3$. That gives: $x – \pi/4 = \pi/3 \rightarrow x = \pi/3 + \pi/4 = (7\pi)/12$

2. $\sin (x – \pi/4) = \sin (\pi – \pi/3)$. That gives $\rightarrow x = \pi – \pi/3 + \pi/4 = (11\pi)/12$

These 2 end points $(7\pi/12)$ and $(11\pi/12)$ divide the unit circle into 2 arc lengths.
To find the sign status of $f(x)$, take point π as check point. We get:

$f(\pi) = \sin(\pi - \pi/4) - \sqrt{3}/2 = \sin(3\pi/4) - \sqrt{3}/2 = \sqrt{2}/2 - \sqrt{3}/2 < 0$.

Therefor, f(x) is negative ($f(x) < 0$) inside the arc length $((11\pi)/12, (2\pi + (7\pi)/12)$. That is the solution set. See Figure 5.

Figure 5

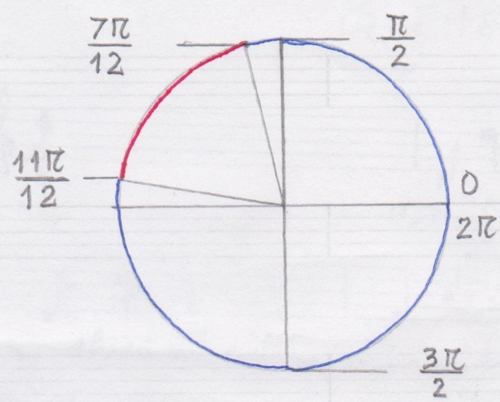
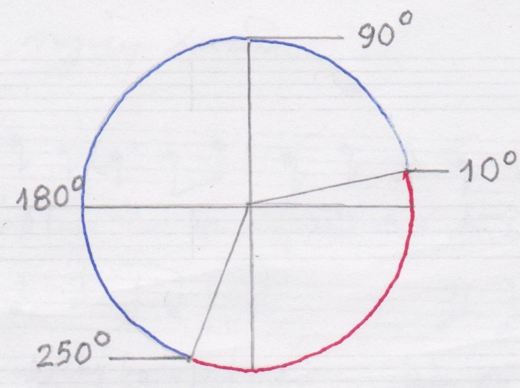


Figure 6



Check.

$x = 3\pi/2 \rightarrow f(x) = \sin(5\pi/4) - \sqrt{3}/2 < 0$. Proved
$x = 0 \rightarrow f(x) = \sin(-\pi/4) - \sqrt{3}/2 < 0$. Proved

Example 6. Solve $\cos(x + 50^\circ) > 0.5$ $(0, 360^\circ)$

Solution. General form: $f(x) = \cos(x + 50^\circ) – 0.5 > 0$

First solve $f(x) = \cos(x + 50) – 0.5 = 0$. Calculator gives: $0.5 = \cos(\pm 60^\circ)$. The trig circle gives 2 solutions:

1. $x + 50 = 60$. That gives $\rightarrow x = 60 – 50 = 10^\circ$
2. $x + 50 = - 60$. That gives $\rightarrow x = - 60 – 50 = - 110^\circ$, or $x = 250^\circ$

To find the sign status of f(x), take as check point the point $x = 90^\circ$. We get:
$f(90) = \cos 140^\circ – 0.5 = - 0.76 – 0.5 < 0$. Then, f(x) is negative (< 0) inside the arc length $(10^\circ, 250^\circ)$. The solution set $f(x) > 0$ is the interval **$(250^\circ, 370^\circ)$.** Figure 6.

Check.

$x = 180^\circ \rightarrow f(180) = \cos 230^\circ – 0.5 = - 0.64 – 0.5 < 0$. Proved
$x = 0 \rightarrow f(0) = \cos 50^\circ – 0.5 = 0.64 – 0.5 > 0$. Proved

NOTE. On the trig circle, we color red the arc length where f(x) > 0. We color blue the arc length where f(x) < 0. This way of coloring will help later when solving complex trig inequalities.

Example 7. Solve tan x < 3/4. (0, 2π)

Solution. On the tangent axis, get AT = 0.75. Calculator give arc AM1 = 36º87 and arc AM2 = 180º + 36º87 = 216º87.

By considering the unit circle, we see that F(x) = tan x - 3/4 < 0 when arc x varies inside the arc length (- 90º, 36º87).

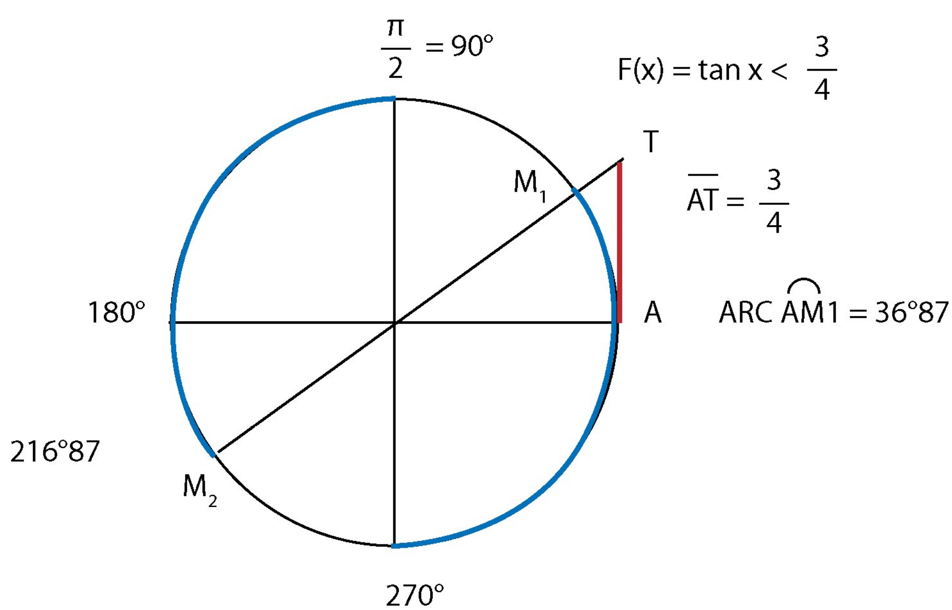
The answer is: 270º < x < (360º + 36º87 = 396º87)
The extended answer is 270º + k180º < x < 396º87 + k180º

For k =1, there is another arc length (90º, 216º87) that makes the inequality true.

Finally, the solution set of the inequality **F(x) < 0,** for (0, 360º) is 2 open intervals **(90º, 216º87)** and **(270º, 396º87)**

Note. We can use the point A(0) as check point. We get: F(0) = tan (0) – 3/4 = 0 – 3/4 < 0. It is true, then, the point A is located on the solution set. See Figure 7.

Figure 7



Example 8. Solve tan (x – 45º) > √3/3

Solution. General form: f(x) = tan (x – 45º) - √3/3 > 0.

First solve f(x) = 0, that gives → tan (x – 45º) = √3/3. The trig Table of Special Arcs and the unit circle give 2 solutions:

1. tan (x – 45) = tan 30º. That gives → x – 45º = 30º → x = 30 + 45 = 75º
2. tan (x – 45º) = tan (210º). That gives → x = 210 + 45 = 255º

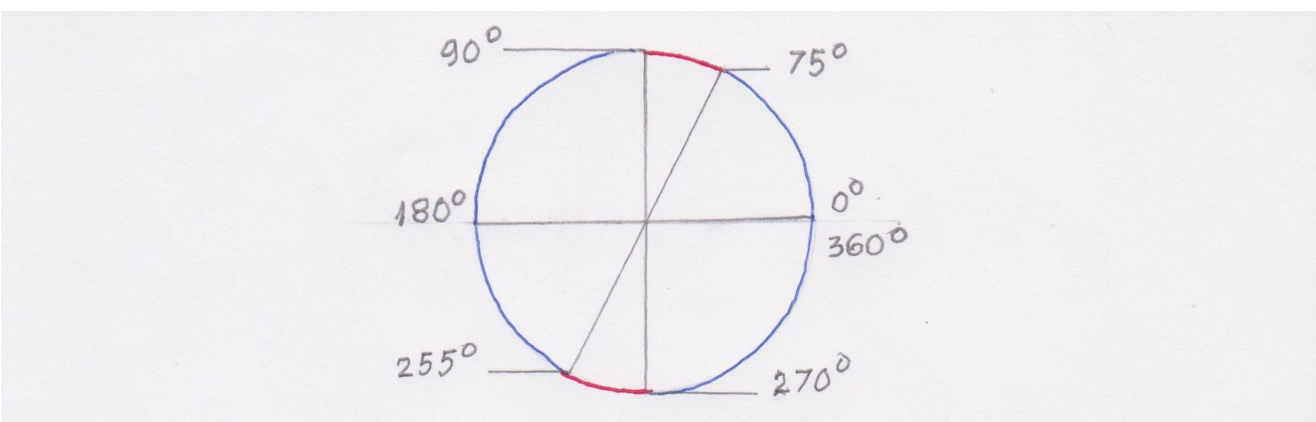
The function f(x) is undefined when x = (ЛT/2) and x = (3ЛT/2). The sin-axis and the 2 end points (75) and (255) divide the unit circle into 4 arc lengths, or intervals:

Inside interval (75º, 90º), f(x) is positive (> 0). Color red → Solution set
Inside interval (90º, 255º), f(x) is negative (< 0). Blue
Inside interval (255º, 270º), f(x) is positive (> 0). Red → Solution set
Inside interval (270º, 435º), f(x) is negative (< 0). Blue. See Figure 8

Check.

x = 0. This gives --> f(0) = tan (-45) - √3/3 < 0. Proved
x = 180º. This gives --> f(180) = tan (135) - √3/3 = - 1 - √3/3 < 0. Proved
x = 100º. This gives → f(100) = tan (55) - √3/3 = 1.43 – 0.58 > 0. Proved.

Figure 8



Example 9. Solve cos 3x > 0 (0, 2Л)

Solution. General form: f(x) = cos 3x > 0

First solve f(x) = 0 to find the 6 end points. On the trig circle, cos 3x = 0 when:

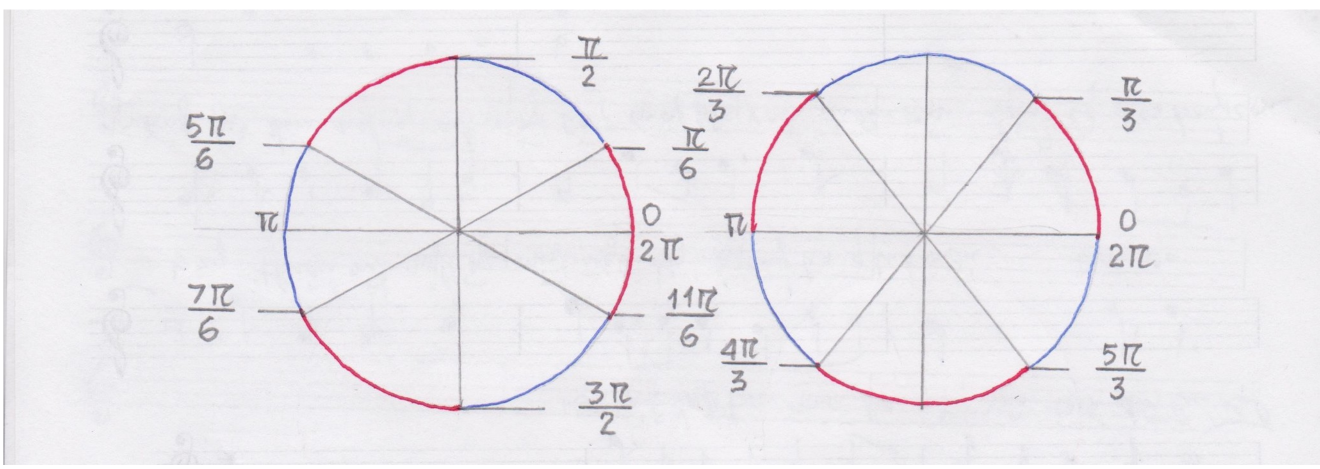
1. cos 3x = cos Л/2, that gives: 3x = Л/2 + 2kЛ → x = Л/6 + (2kЛ)/3
2. cos 3x = cos (3Л/2), that gives: 3x = 3Л/2 + 2kЛ → x = Л/2 + (2kЛ)/3.

For k = 0, k = 1, k = 2, there are 6 end points at: Л/6, Л/2, 5Л/6, 7Л/6, 3Л/2, and 11Л/6. There are 6 equal arc lengths.

To find the sign status of f(x) inside the interval (11Л/6, 13Л/6), select point 0 as check point. We get f(0) = cos 0 = 1 > 0. Therefor, f(x) > 0 in this interval. Color it red. Color the next 5 arc lengths. We get from point Л/6: blue, red, blue, red, blue.
The solution set are the 3 red open intervals:

(Л/2, 5Л/6) and **(7Л/6, 3Л/2)** and **(11Л/6, 13Л/6)**. See Figure 9

Figure 9 Figure 10



Example 10. Solve sin 3x < 0 (0, 2Л)

Solution. General form: f(x) = sin 3x < 0

First solve f(x) = sin 3x = 0 to find the 6 endpoints and the six equal arc lengths.
On the unit circle sin 3x = 0 when:

1. sin 3x = sin 0. This gives: 3x = 0 + 2kπ → x = (2kπ)/3
2. sin 3x = sin π. This gives: 3x = π + 2kπ → x = π/3 + (2kπ)/3
3. sim 3x = sin 2π. This gives: 3x = 2π + 2kπ → x = (2π)/3 + (2kπ)/3

For k = 1, k = 2, k = 3, there are 6 end points at: 0, π/3, 2π/3, π, 4π/3, 5π/3.
There are 6 equal arc lengths.
Select the check point x = π/2. We get: f(π/2) = sin (3π)/2 = -1 < 0. Therefor, f(x) is negative (< 0) inside the interval (π/3, 2π/3). Color it blue, and color the 5 others.

The solution set for f(x) < 0 are the 3 blue open intervals:
(π/3, 2π/3), and **(π, 4π/3)**, and **(5π/3, 2π).** See Figure 10

Example 11. Solve cos 2x < 0 (0, 2π)

Solution. f(x) = cos 2x < 0 (0, 2π)

First, solve f(x) = cos 2x = 0 to find the 4 end points and 4 equal arc lengths.
On the unit circle, cos 2x = 0 when:

1. cos 2x = cos π/2, that gives 2x = π/2 + 2kπ → x = π/4 + kπ
2. cos 2x = cos (3π)/2, that gives 2x = 3π/2 + 2kπ → x = (3π)/4 + kπ.

There are 4 end points at: π/4, (3π)/4, (5π)/4 and (7π)/4. There are 4 equal arc lengths. Find the sign status of arc length (π/4, 3π/4). Select point (π/2) as check point. We get: f(π/2) = cos π = -1 < 0. Therefor, f(x) < 0 inside this interval. Color it blue, and color the other 3 arc lengths.

The solution set are the blue intervals: **(π/4, 3π/4)** and **(5π/4, 7π/4).** See Figure 11

Figure 11

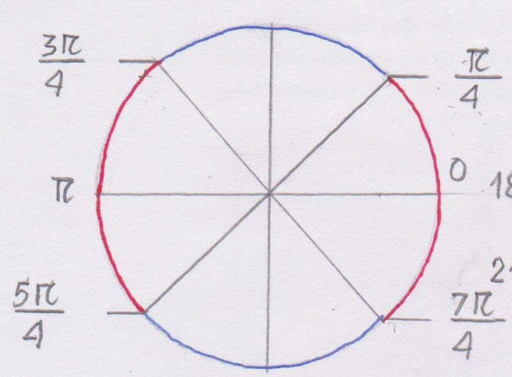
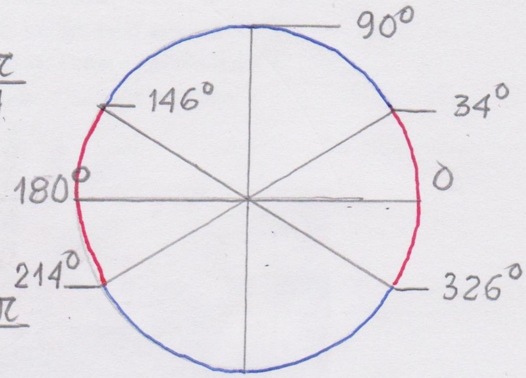


Figure 12



Example 12. Solve cos 2x < 0.374

Solution. General form f(x) = cos 2x – 0.374 < 0 (0, 360º)

First solve f(x) = 0 to find the 4 end points and the 4 arc lengths. Calculator gives 0.374 = cos (± 68º). There are 2 solutions:

1. cos 2x = cos (68º). This gives: 2x = 68º + k360º → x = 34º + k180º
2. cos 2x = cos (- 68º). This gives: 2x = - 68º + k360º → x = - 34º + k180º

For k = 1, there are 4 end points at: 34º, 146º, 214º, 326º (or – 34º).
There are 4 arc lengths. To find the sign status of f(x) inside the arc length (34º, 146º), take the check point x = 90º. We get: f(90) = cos 180 – 0.374 = - 1 – 0.374 < 0.
Therefor, f(x) < 0 in this interval. Color it blue, and color the 3 other arc lengths.
The solution set are the 2 blue intervals: **(34º, 146º)** and **(214º, 326º).** See Figure 12.

Example 13. Solve cos (2x – 65º) < 0.643

Solution. General form: f(x) = cos (2x – 65º) – 0.656 < 0 (0, 360º)

Solve f(x) = 0 to get the 2 end points. Calculator gives: 0.656 = cos (± 49º). We have:

1. 2x – 65 = 49. That gives 2x = 49 + 65 = 114 + k360 → x = 57º + k180º
2. 2x – 65 = - 49. That gives: 2x = 16 + k360 → x = 8º + k180º

For k = 1, we get 4 end points at: 8º, 57º, 188º, and 237º and 4 arc lengths.
Select O as the check point. We get: f(0) = cos (- 65) – 0.656 = 0.42 – 0.656 < 0.
Therefor, the 2 blue arc lengths: (237º, 368º) and (57º, 188º) are the solution set. See Figure 13.

Figure 13

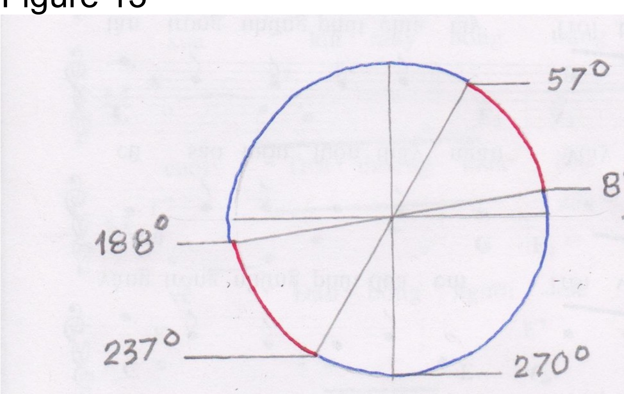
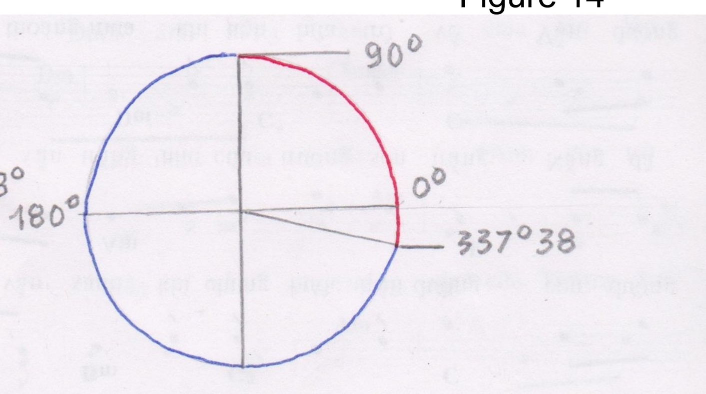


Figure 14



Example 14. Solve $2\sin x + 3\cos x < 2$ $(0, 360º)$

Solution. Divide both side by 2, we get: $\sin x + (3/2)\cos x < 1$.

Call t the arc that $\tan t = 3/2$. This gives $t = 56º31$ and $\cos t = 0.55$.

$\sin x + (\sin t/\cos t).\cos x < 1$
$\sin x.\cos t + \sin t.\cos x < \cos t = 0.55 = \sin (33º69) = \sin (146º31)$

Using the trig identity: $\sin (a + b) = \sin a.\cos.b + \sin b.\cos a$, we get:

$\sin (x + 56º31) = \sin (33º69) = \sin (146º31) \rightarrow$ This leads to 2 equations:

1. $x + 56º31 = 33º69$. This gives $x = - 22º62$, or $x = 337º38$ (co-terminal)
2. $x + 56º31 = 146º31$. This gives $x = 146º31 – 56º31 = 90º$

There are 2 end points at $x = 90º$ and $x = (337º38)$, and 2 arc lengths. To find the sign status of $f(x) = 2\sin – 3\cos x – 2< 0$, select the point (180º) as check point. We get: $f(180º) = 0 – 3 – 2 < 0$. Therefore, f(x) is negative in this interval (90º,337º38). Color it blue and color the other arc length red.

The solution set of $f(x) < 0$ is the open interval **(90º, 337º38).** See Figure 14.

NOTE. Solving some complex trig inequalities can be resulted to solving one basic trig inequality.

Exercise 15. Solve $\sin^3 x + \cos^3 x > 0$ $(0, 2\pi)$

Solution. Using algebraic identity: $a^3 + b^3 = (a + b) (a^2 – ab + b^2)$, we get:
$F(x) = (\sin x + \cos x).(\sin^2 x– \sin x.\cos x + \cos^2 x)$
$F(x) = f(x).g(x) = (\sin x + \cos x)(1 – \sin x.\cos x) > 0$.

Since $g(x) = 1 – \sin x.\cos x$ is always positive regardless of x, therefore, the sign status of F(x) is the sign status of f(x).

Solve $f(x) = (\sin x + \cos x) = 0$. Use trig identity $(\sin a + \cos a) = \sqrt{2}\cos (a – \pi/4)$.

$f(x) = \sqrt{2}\cos (x – \pi/4) = 0$. This gives 2 solutions:
$x – \pi/4 = \pi/2$ --> that gives $x = \pi/2 + \pi/4 = 3\pi/4$
$x – \pi/4 = 3\pi/2 \rightarrow$ that gives $x = 3\pi/2 + \pi/4 = 7\pi/4$

There are 2 end points at $(3\pi/4)$ and $(7\pi/4)$, and 2 arc lengths. Find the sign status of f(x) by using point $(\pi/2)$ as check point.

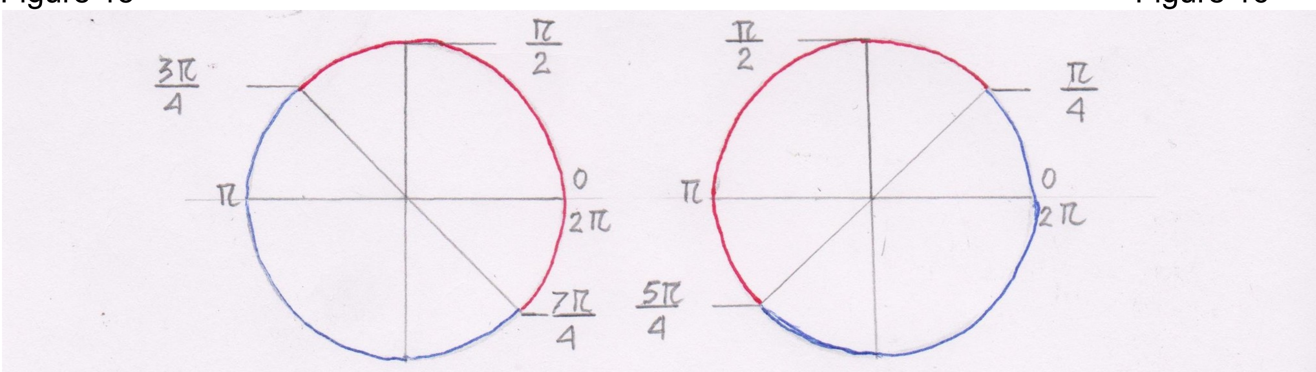
We have: $f(\pi/2) = \cos(\pi/2) - \pi/4) = \cos \pi/4 > 0$.

Therefore, f(x) is positive (> 0) in the interval $(-\pi/4, 3\pi/4)$. Color it red and the rest blue. On the unit circle, the solution set of $F(x) > 0$ (red) is the open interval **$(-\pi/4, 3\pi/4)$.** See Figure 15.

Check $\sin^3 x + \cos^3 x > 0$

$x = \pi$. This gives: $F(x) = 0 - 1 < 0$ (proved)
$x = 0$. This give $F(x) = 0 + 1 > 0$ (proved)

Figure 15 Figure 16



Exercise 16. Solve $\sin^3 x - \cos^3 x \le 0$

Solution. Using algebraic identity: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we get:

$F(x) = f(x).g(x) = \sin^3 x - \cos^3 x = (\sin x - \cos x)(1 + \sin x.\cos x) \le 0$.(Period 2π)

Since the function $g(x) = (1 + \sin x.\cos x)$ is always positive regardless of x, the sign status of F(x) is the one of f(x).

Solve $f(x) = (\sin x - \cos x) = \sqrt{2}.\sin(x - \pi/4) = 0$. Using the trig identity: $\sin a - \sin b = \sqrt{2}.\sin(a - \pi/4)$, we get 3 equations:

$(x - \pi/4) = \sin 0$. This gives: $x - \pi/4 = 0 \rightarrow x = \pi/4$
$(x - \pi/4) = \sin \pi$. This gives: $x - \pi/4 = \pi \rightarrow x = 5\pi/4$
$(x - \pi/4) = \sin 2\pi$. This gives: $x = 2\pi + 2\pi + \pi/4 = 9\pi/4$ or $\pi/4$

There are 2 end points at $(\pi/4)$ and at $(5\pi/4)$, and 2 arc lengths.
Find the sign status of f(x) by selecting point $(\pi/2)$ as check point. We have $f(\pi/2) = \sin(\pi/2 - \pi/4) = \sin \pi/4 > 0$. Then, f(x) is positive (> 0) inside interval $(\pi/4, 5\pi/4)$.

Color the arc length red and the rest blue.

On the unit circle, we see that the solution set of $F(x) \leq 0$ (blue) is the closed interval **[5Л/4, 9Л/4)**. See Figure 16.

REMARKS

1. A few simple rules for endpoints and arc lengths of basic trig functions in x.
- For f(x) = sin x and f(x) = cos x, and similar, there are 2 endpoints and 2 arc lengths for (0, 2Л)
- For f(x) = sin 2x & f(x) = cos 2x, and similar, there are 4 endpoints and 4 arc lengths for (0, 2Л)
- For f(x) = sin 3x & f(x) = cos 3x, and similar, there are 6 endpoints and 6 arc lengths (0, 2Л)
- For f(x) = tan x there are 1 endpoint, one discontinuation and two arc lengths for (-Л/2, Л/2)
- For f(x) = cot x, there are 1 endpoint, one discontinuation, and two arc lengths for (0, Л)

2. Advantages of the Nghi Nguyen Method

- In the Sign Chart Method, the 2 extremities (0, 2Л) of the sign chart are separate that makes the chart confusing. In the new method there is continuation that shows the periodic property of trig functions.

- In the sign chart, all values of x are placed on the first line of the chart. In case of complex trig inequalities, such as $F(x) = f(x).g(x) \leq 0$ (or ≥ 0) or $F(x) = f(x).g(x)h(x) \leq 0$, (or ≥ 0), the high number of x makes the chart stuffy and confusing. In the new method, the number of x depends only on, in each step, the solving of one basic inequality.

- In the sign chart method, it is hard to find the sign status of each presented basic trig function. In the new method, the innovative technique, using the check point method, makes the sign status identification very convenient.

(Authored by Nghi H Nguyen, Sept. 01, 2021)