

# A simple formula for solving quadratic equations using function evaluation.

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## Abstract

This paper presents a simple formula for solving quadratic equations using function evaluation. The formula is simple to remember and use compared with the existing methods of solving quadratic equations. In addition, through this formula, students learning math will be presented with methods of solving equations from yet another points of view. The paper is only of educational value and does not claim to be of any novel invention.

**Key words** Quadratic equations, math education, roots of equations, polynomial equations, algebra

## 1. Introduction

The subject of learning quadratic equations is a useful mathematical topic that is common place in algebra and because of its application as a workable solution tool in mathematical problems featuring in science and technology. (Graf *et al.*, 2018). The discovery of the formula of solutions to quadratic equations with arbitrary coefficients marked important milestone following the search for solutions dating as far back as the Old Babylonian Period 2000-1600 B.C. (Loh, 2019). Early methods of solving quadratic equations usually involved numerical procedures, that essentially used competing squares (Kats, 2007). In the early times negative numbers were unknown and equations that are reduced to such forms as  $x-y = b$  and  $xy=c$  require that  $x$  be always greater than  $y$ . In the 17<sup>th</sup> Century Descartes developed the completing square method to a more general quadratic formula allowing both negative and positive coefficients and roots. For the quadratic equations expressed in terms of arbitrary coefficients  $ax^2 +bx +c=0$ , the quadratic formula for the solution  $x$  was expressed as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula is experienced by early math learners as a somehow complicated algebra that needs to be remembered. Methods such as completing squares are included in learning of the methods of solving quadratic solution as less memory demanding procedure but their inclusion is not always standard practice like the standard formula involving radicals. The study of quadratic equations acts as a gateway to more advanced study of algebra and is a topic area that challenges many students (O'Connor, et al., 2016). Some curricula emphasize easier methods of solving quadratics that involve factoring which tend to reinforce the belief that all quadratic equations

with integer coefficients may be factored into rational roots which according to Bosse and Nandakumar (2005) has only 15% probability of occurrence. Students limited to learning solution by factoring may experience difficulty working around quadratic equation appearing in forms that is not conducive to factoring (Planel, 2015).

There were four basic methods of solving quadratic equations throughout history: The square root method, completing square, quadratic formula and factorization (Gunar, and Uygun, 2018). The square root method works for equations that appear in a form  $x^2 + c = 0$  in which the solutions can be found by taking square roots of the constant. Completing squares involves a procedure in which for the equation  $x^2 + bx + c$ , the square is completed so that the equation is transformed into  $(x+b/2)^2 = (b^2-4c)/4$  after which square root method is applied to find the roots. The quadratic formula uses the formula developed containing the arbitrary coefficients and can be used directly once the formula is memorized. Factorisation works in general for quadratic equations with integer coefficients that can be factored in to rational numbers. The given quadratic equation  $x^2 + bx + c = 0$  is factored into  $(x-r)(x-s)$  in which the roots  $r$  and  $s$  satisfy the requirement  $r+s = -b$  and  $r \times s = c$ . Graphical methods are also employed to increase students' understanding of quadratic equations in which the plot of the quadratic function shows cases involving two roots, a single root and no real root (complex roots) and linking these to the discriminant (Barnes, 1991).

Other methods and their modifications have also evolved over time. The Diagonal Sum Method (Nguyen, 2019<sup>a</sup>) enumerates possible roots of the quadratic equation  $ax^2 + bx + c$  using pairs  $(C_1/A_1, C_2/A_2)$  in which  $C_1C_2 = c$  and  $A_1A_2 = a$ . Out of the possible pairs, the one in which the diagonal sums, i.e.,  $C_1A_2 + C_2A_1 = -b$  are chosen as the roots. The Bluma method (Clark and Milman, 2019) transforms quadratic equation  $ax^2 + bx + c$  by multiplying the constant term  $c$  by the coefficient of the  $x$ -squared term  $a$  and solves the resulting equation  $x^2 + bx + ac = 0$  by factoring and the resulting solution is transformed back to the original by dividing by the factor  $a$ . The authors (Clark and Milman, 2019) provide a detailed proof this method. The New Transforming Method (Nguyen, 2019<sup>b</sup>) essentially carries out a Bluma with the added job of enumeration of the pairs of possible root factors until they sum to  $-b$ . The advantages of the new Transforming Method are: simple, fast, systematic, no guessing, no factoring by grouping and no solving binomials.

## 2. The new formula and its derivation

Below is presented the formula for solving quadratic equations as well as two alternative procedures for deriving this formula. For a general quadratic equation of the form:

$$ax^2 + bx + c = 0 \tag{1}$$

The roots  $x_r$  of the above equation are given by:

$$x_r = Z \pm \sqrt{\frac{-f(Z)}{a}} \quad ; \quad Z = \frac{-b}{2a} \quad (2)$$

In short Equation (2) gives the solution in terms of the transformed variable Z and its functional value only. If the quadratic equation is in reduced form, i.e.,

$$x^2 + bx + c = 0 \quad (3)$$

The solution would further simplify to:

$$x_r = Z \pm \sqrt{-f(Z)} \quad ; \quad Z = \frac{-b}{2} \quad (4)$$

This formula (Equation 4) would be quite easy to remember as it does not contain any of the symbols of the original equation. Now the derivation of this formula is presented below using two alternative procedures.

#### Method 1:

Consider the quadratic equation given by Equation (1), i.e.,

$$f(x) = ax^2 + bx + c = 0 \quad (1)$$

Define the new variable Z such that:

$$x = Z + m [f(x) - f(Z)] \quad (5)$$

Where Z is the new variable, f(x) and f(z) are the functional values evaluated At x and Z respectively, and m is an arbitrary constant. At the root  $x = x_r, y = 0$ , Equation (5) simplifies to:

$$x_r = Z_r - m f(Z_r) \quad (6)$$

Substituting the x value of Equation (6) into Equation (1) gives:

$$aZ_r^2 - 2 a m Z_r f(Z_r) + am^2 [f(Z_r)]^2 + bZ_r - bm f(Z_r) + c = 0 \quad (7)$$

Rearranging Equation (7) gives the following:

$$aZ_r^2 + bZ_r + c - m f(Z_r)[2 a Z_r + b] + am^2 [f(Z_r)]^2 = 0 \quad (8)$$

Since  $aZ_r^2 + bZ_r + c = f(Z_r)$ , Equation 8 further reduces to:

$$f(Z_r) - m f(Z_r)[2 a Z_r + b] + am^2[f(Z_r)]^2 = 0 \quad (9)$$

Choosing the  $Z_r$  value so that:

$$Z_r = \frac{-b}{2a} ; \quad 2 a Z_r + b = 0 \quad (10)$$

The above choice of  $Z_r = -b/2a$  will eliminate the middle term of Equation (9) which now will reduce to:

$$f(Z_r) + am^2[f(Z_r)]^2 = 0 \quad (10)$$

Factoring out  $F(Z_r)=0$  which cannot be true since  $Z_r = -b/2a$  is already fixed and solving for the unknown  $m$  in Equation (10) will give :

$$m = \pm \sqrt{\frac{-1}{af(Z_r)}} \quad (11)$$

Substituting the  $m$  value from Equation (11) above into Equation (6) gives:

$$x_r = Z_r - m f(Z_r) = Z_r \pm f(Z_r) \sqrt{\frac{-1}{af(Z_r)}}$$

Simplifying further;

$$x_r = Z_r \pm \sqrt{\frac{-1[f(Z_r)]^2}{af(Z_r)}} = Z_r \pm \sqrt{\frac{-f(Z_r)}{a}} ; \quad Z_r = \frac{-b}{2a}$$

Which completes the proof of the formula given by Equation (2)

### Method 2:

With hindsight of the derivation of the formula from Method 1 above, let us try to evaluate the quadratic function at  $x = -b/2a$ , i.e.,

$$f \left( Z_r = \frac{-b}{2a} \right) = a \left( \frac{-b}{2a} \right)^2 + b \left( \frac{-b}{2a} \right) + c = - \left( \frac{b^2 - 4ac}{4a} \right) \quad (12)$$

Considering the well-known solution of the quadratic equation

$$x_r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b}{2a} \pm \sqrt{\frac{1}{a} \left( \frac{b^2 - 4ac}{4a} \right)} \quad (13)$$

Substitution the expression for  $f(-b/2a)$  from Equation (12) to equation (13) gives;

$$x_r = \frac{-b}{2a} \pm \sqrt{\frac{-f\left(\frac{-b}{2a}\right)}{a}} = Z + \sqrt{\frac{-f(Z)}{a}} \quad ; \quad Z = \frac{-b}{2a}$$

Which completes the proof of the formula given by Equation (2)

### 3. Worked examples

#### Example 1

Consider the quadratic equation:

$$f(x) = 3x^2 - 8x - 35 = 0$$

In the above equation ,  $a = 3$  and  $b = -8$ . Calculate  $Z$  and  $f(Z)$

$$Z = \frac{-b}{2a} = \frac{-(-8)}{2(3)} = \frac{4}{3}$$

$$f(z) = 3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) - 35 = \frac{-121}{3}$$

The solution of the above equation using Equation (2) will be:

$$x_r = Z \pm \sqrt{\frac{-f(Z)}{a}} = \frac{4}{3} \pm \sqrt{-\frac{1}{3} \left( \frac{-121}{3} \right)} = \frac{4}{3} \pm \frac{11}{3}$$

The solutions are therefore,  $x_r = 5$  and  $x_r = -7/3$ .

#### Example 2

Consider the simplified equation :

$$f(x) = x^2 - 4x + 3 = 0$$

In the above equation ,  $b = -4$ . Calculate  $Z$  and  $f(Z)$

$$Z = \frac{-b}{2} = \frac{-(-4)}{2} = 2$$

$$f(z) = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

The solution of the above equation using Equation (4) will be:

$$x_r = Z \pm \sqrt{-f(Z)} = 2 \pm \sqrt{-(-1)} = 2 \pm 1 = \{3, 1\}$$

The solution  $x = \{3, 1\}$  can be easily worked out also by the method of factoring.

Example 3:

Finally consider the quadratic equation:

$$f(x) = 2x^2 + 3x - 5 = 0$$

In the above equation,  $a = 2$  and  $b = 3$ . Calculate  $Z$  and  $f(Z)$

$$Z = \frac{-b}{2a} = \frac{-(3)}{2(2)} = \frac{-3}{4}$$

$$f(z) = 2\left(\frac{-3}{4}\right)^2 + 3\left(\frac{-3}{4}\right) - 5 = \frac{-49}{8}$$

The solution of the above equation using Equation (2) will be:

$$x_r = Z \pm \sqrt{\frac{-f(Z)}{a}} = \frac{-3}{4} \pm \sqrt{-\frac{1}{2}\left(\frac{-49}{8}\right)} = \frac{-3}{4} \pm \frac{7}{4}$$

The solutions are, therefore,  $x_r = 1$  and  $x_r = -5/2$ .

#### 4. Conclusion

The formula presented in this paper helps in easily memorizing the solution to quadratic equations in which the solution is worked out only in terms of the transformed variable and its

functional value. The derivation of the formula and its application through the worked examples shown in this paper provides another point of view to students learning math in understanding the development of methods for solving polynomial equations.

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